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Relativistic Quark-Antiquark Bound State Problem with Spin-dependent Interactions in Momentum Space

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Abstract

The work described in this paper is the first step toward a relativistic three-quark bound-state calculation using a Hamiltonian consistent with the Wigner-Bargmann theorem and macroscopic locality. We give an explicit demonstration that we can solve the two-body problem in momentum space with spin-dependent interactions. The form of the potential is a combination of linear+Coulomb+spin-spin+spin-orbit+tensor, which includes confinement and is of the general form consistent with rotation, space-reflection and time-reversal invariance. Comparison is made with previous calculations using an alternate technique and with the experimental meson mass spectrum. The results obtained suggest that the model is realistic enough to provide a two-body basis for the three quark baryon problem in which the Poincaré group representation is unitary and cluster separability is respected.

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I. INTRODUCTION

The work described in this paper is the first step toward a relativistic three-fermion calculation. For light quark systems, wherein mesons are described by quark-antiquark dynamics and baryons by three quark dynamics, one calculable approach is the relativistic potential model. Covariant approaches based on the Bethe-Salpeter equation [1] appear to require some form of reduction to a soluble equation, e.g., Salpeter's instantaneous approximation [2], ladder approximations in Euclidean space [3] or a relativistic equation in which one of the quarks is restricted to its mass shell [4]. Each of these reductions can be argued to have its own strengths and weaknesses. There is currently no truly satisfactory solution of the Bethe-Salpeter equation for bound states of two fermions, let alone three. Therefore, we resort to a potential model with a model Hamiltonian consistent with the principle of relativity and macroscopic locality [5] to solve the three light-quark bound-state problem. The three-body mass operator (Hamiltonian) can be defined using the Bakamjian-Thomas construction [6]. Unfortunately such a relativistic Hamiltonian is already very difficult to solve [5,7]. One feasible approach is to solve the Faddeev equations in momentum space using appropriate two-body wave functions as a basis [7,8]. This necessarily requires an accurate calculation of such a two-body wave function basis. We have chosen the collocation method to solve the two-body equation rather than variational techniques for this reason. Consequently, it is necessary to first solve the quark-antiquark problem accurately in momentum space (because of the difficulty arising from the square root operator in coordinate space using the collocation method) with general spin-dependent interactions. This has not previously been done. Previous work in momentum space has been limited to spin-independent calculation [7,9]. In the present work we attempt to ensure that the two-body basis is optimal, we fit the meson masses (i.e., quark-antiquark systems) with a parametrized interaction in momentum space including a full range of spin-dependent interactions.

In section II we describe the relativistic two-body Hamiltonian in terms of a linear confinement term , a Coulomb term, and various spin-dependent pieces. We emphasize that this is not a simple nonrelativistic reduction of an effective one-gluon-exchange potential since the coefficients of the various terms in the potential will be constrained by phenomenological considerations alone. The various terms making up the potential are used because they are invariant under rotations, space-reflection and time-reversal. To treat singularities in the spin-dependent interactions a form factor is introduced for the quark-gluon vertex. Since the nonperturbative quark-gluon vertex is in general momentum-dependent, this is a natural thing to include. Numerical solutions are obtained for the model of reference [10] using partial wave expansions of the various interactions given in section II. Results for various parameter choices including those of reference [10] are presented in section III. These results show that the light meson spectrum can be well described by such a model. Planned future directions involving other properties of the meson system, such as electromagnetic form factors, and the extension to relativistic three-fermion systems are given in section IV.

II. FORMALISM

In our relativistic potential model, the Hamiltonian H is the sum of a relativistic kinetic energy operator T and a potential operator V. The kinetic energy operator has the form

$$T = \sqrt{M_1^2 + \mathbf{k}^2} + \sqrt{M_2^2 + \mathbf{k}^2} \ , \tag{1}$$

where M_1, M_2 are the masses of the two quarks and $\mathbf{k}$ is the relative momentum. In what follows, we assume $M \equiv M_1 = M_2$. The potential is the sum of a linear confining potential and a short range spin-dependent interaction:

$$V = V_L + V_S \ . \tag{2}$$

Here we use a linear confining potential $V_L = \sqrt{\sigma} r$ where $\sqrt{\sigma}$ is the string tension and r is the relative coordinate. As a guide to choosing an effective spin-dependent short range interaction we begin by noting that the nonrelativistic reduction of the one-gluon-exchange potential in momentum space has the form [11]

$$\begin{aligned} \langle \mathbf{k}|V_{NR}|\mathbf{k}'\rangle = f_c\alpha_s\frac{1}{2\pi^2}\Bigg\{ & \frac{1}{(\mathbf{k}'-\mathbf{k})^2} - \frac{1}{6M^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + \frac{3}{4M^2}\frac{1}{(\mathbf{k}'-\mathbf{k})^2} i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\times\mathbf{k}' \\ & + \frac{1}{4M^2}\left[(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{q}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{q}}) - \frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right]\Bigg\}, \end{aligned} \tag{3}$$

where α_s is the strong-interaction fine-structure constant, f_c is the color factor (which is $-4/3$ for quark-antiquark and $-2/3$ for quark-quark), $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ are the Pauli matrices and $\mathbf{q} \equiv \mathbf{k}' - \mathbf{k}$ is the momentum transfer. It is readily seen that the four terms in the curly brackets represent the Coulomb potential, the spin-spin, spin-orbit and tensor interactions respectively. These are easily recognized in coordinate space representation:

$$\begin{aligned} \langle \mathbf{r}|V_{NR}|\mathbf{r}'\rangle = \delta(\mathbf{r}-\mathbf{r}')f_c\alpha_s\Bigg\{ & \frac{1}{r} - \frac{2\pi}{3M^2}\delta(\mathbf{r})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) - \frac{3}{2M^2r^3}\mathbf{L}\cdot\mathbf{S} \\ & - \frac{3}{4M^2r^3}\left[(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}) - \frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right]\Bigg\}, \end{aligned} \tag{4}$$

where $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$ is the total spin operator and $\mathbf{L}$ is the orbital angular momentum operator. The spin-spin interaction includes a delta function of $\mathbf{r}$, (i.e., it is a contact interaction), and so we introduce a Gaussian form factor $\exp(-\frac{1}{2}\Lambda^2\mathbf{q}^2)$ at the quark-gluon vertex as in Ref. [10]. The variable Λ can be interpreted as the size of the quark. In Ref. [12] a form factor $1/(\mathbf{q}^2 + \beta^2)$ in which β^{-1} is the effective quark size is used to eliminate the singularity. Since the one-gluon-exchange potential (Eq. 3) is derived via a nonrelativistic reduction (and so can not represent a reasonable interaction in a relativistic calculation) and in any case is really an effective interaction, we are completely free to vary the relative strengths of these interactions. In particular it is well known that the phenomenological strength of the spin-orbit interaction is much weaker than that predicted by the nonrelativistic reduction of the one-gluon-exchange potential [13]. Hence we use multiplying factors $C_L, C_C, C_{SS}, C_{LS}, C_T$ so that the strength of every term can be varied to fit the meson data. The general form of the potential that we used in our calculation then has the form

$$\begin{aligned} \langle \mathbf{k}|V|\mathbf{k}'\rangle = C_L V_L + \frac{f_c\alpha_s}{2\pi^2}e^{-\Lambda^2\mathbf{q}^2}\Bigg\{ & \frac{C_C}{(\mathbf{k}'-\mathbf{k})^2} \\ & - \frac{C_{SS}}{6M^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \end{aligned}$$

$$+\frac{3C_{LS}}{4M^2}\frac{1}{(\mathbf{k}'-\mathbf{k})^2}i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\times\mathbf{k}'$$
$$+\frac{C_T}{4M^2}\left[(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{q}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{q}})-\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right]\Bigg\}. \tag{5}$$

Note that the only potentials that respect rotation invariant, space-reflection and time-reversal which are not included in our potential are $(\boldsymbol{\sigma}_1\cdot\mathbf{L})(\boldsymbol{\sigma}_2\cdot\mathbf{L})$ and $(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{L}$. The latter term becomes significant only if $l\neq 0$ and one of the quarks is much heavier than other.

To do the calculations in momentum space we have to use a partial-wave expansion for the potentials in Eq. (5). The difficulty with a momentum-space formulation for the linear potential was solved in Refs. [14,15]. The following are the results for the partial-wave expansion of the potentials in momentum space.
For potentials depending on $|\mathbf{q}|$ only:

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=\tilde{V}(|\mathbf{q}|)\ , \tag{6a}$$

we find (using $\mu\equiv\hat{\mathbf{k}}'\cdot\hat{\mathbf{k}}$)

$$\langle klsjm|V|k'l's'j'm'\rangle=\delta_{ll'}\delta_{ss'}\delta_{jj'}\delta_{mm'}$$
$$\times 2\pi\int_{-1}^{1}\tilde{V}(|\mathbf{q}|)P_l(\mu)d\mu\ . \tag{6b}$$

For the spin-spin interaction:

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=\tilde{V}(\mathbf{q})\mathbf{S}_1\cdot\mathbf{S}_2\ , \tag{7a}$$
$$\langle klsjm|V|k'l's'j'm'\rangle=\delta_{ll'}\delta_{ss'}\delta_{jj'}\delta_{mm'}$$
$$\times\frac{1}{2}[s(s+1)-s_1(s_1+1)-s_2(s_2+1)]$$
$$\times 2\pi\int_{-1}^{1}\tilde{V}(|\mathbf{q}|)P_l(\mu)d\mu\ . \tag{7b}$$

For the spin-orbit interaction:

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=\tilde{V}(|\mathbf{q}|)i\mathbf{S}\cdot\mathbf{k}\times\mathbf{k}'\ , \tag{8a}$$
$$\langle klsjm|V|k'l's'j'm'\rangle=\delta_{ll'}\delta_{ss'}\delta_{jj'}\delta_{mm'}$$
$$\times\frac{j(j+1)-l(l+1)-s(s+1)}{2}$$
$$\times\frac{2\pi kk'}{2l+1}\int_{-1}^{1}\tilde{V}(|\mathbf{q}|)[P_{l+1}(\mu)-P_{l-1}(\mu)]d\mu\ , \tag{8b}$$

and for the tensor interaction:

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=\tilde{V}(|\mathbf{q}|)q^2\left[3(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{q}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{q}})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right]\ , \tag{9a}$$
$$\langle klsjm|V|k'l's'j'm'\rangle=\delta_{jj'}\delta_{mm'}\delta_{s1}\delta_{s'1}4\sqrt{30}\pi\left\{\begin{array}{ccc} j & 1 & l\\ 2 & l' & 1\end{array}\right\}\sum_{L=0}^{2}\sum_{l''}(-1)^{l'+j+1}(2l''+1)$$

$$\times\sqrt{(2l+1)(2l'+1)(5-2L)C_{2L}^5}\left\{\begin{array}{ccc} l' & 2-L & l'' \\ L & l & 2 \end{array}\right\}$$

$$\times\left(\begin{array}{ccc} l'' & L & l \\ 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc} l'' & 2-L & l' \\ 0 & 0 & 0 \end{array}\right)k^L k'^{2-L}\int \tilde{V}(|\mathbf{q}|)P_{l''}(\mu)d\mu$$

$$=\left\{\begin{array}{l} \delta_{jj'}\delta_{mm'}\delta_{s1}\delta_{s'1}\frac{8l(l+1)-3[l(l+1)-j(j+1)+2][l(l+1)-j(j+1)+1]}{(2l-1)(2l+3)}2\pi \\ \times\int_{-1}^{1}\tilde{V}(q)[(k^2+k'^2)P_l(\mu)-kk'\frac{2l+3}{2l+1}P_{l-1}(\mu)-kk'\frac{2l-1}{2l+1}P_{l+1}(\mu)]d\mu \quad \text{if } l'=l \\ \delta_{jj'}\delta_{mm'}\delta_{s1}\delta_{s'1}\frac{6\sqrt{(L+1)(L+2)}}{(2L+3)}2\pi \\ \times\int_{-1}^{1}\tilde{V}(q)[k'^2P_L(\mu)+k^2P_{L+2}(\mu)-2kk'P_{L+1}(\mu)]d\mu \quad \text{if } l'\neq l, L=\min(l,l') \end{array}\right\}. \quad (9b)$$

In the above equations C_k^n is a binomial coefficient, $P_l(\mu)$ is a Legendre polynomial, $\left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array}\right)$ is a 3-j symbol and $\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}$ is a 6-j symbol [16]. Note that equations (8b) and (9b) are new results. With these two equations, we can do calculations with spin-orbit and tensor interactions for almost any functional form of $\tilde{V}(|\mathbf{q}|)$ in equations (8a) and (9a).

To solve the relativistic wave equation

$$\begin{aligned} &2\sqrt{M^2+\mathbf{k}^2}\Psi_{lsj}(k) \\ &+\sum_{l's'}\int_0^\infty \langle klsjm|V|k'l's'jm\rangle\Psi_{l's'j}(k')k'^2dk' \\ &\quad = E\Psi_{lsj}(k) \end{aligned} \quad (10)$$

we first introduce an auxiliary function $\psi_{lsj}(k)$ given by

$$\Psi_{lsj}(k) = k^{l-1}\psi_{lsj}(k)\ . \quad (11)$$

The power $l-1$ follows from considering the behavior of $\Psi_{lsj}(k)$ as $k\to 0$. The auxiliary wave function is expanded in a complete set of basis functions. In Refs. [12] and [13] the wave functions were expanded in a harmonic oscillator basis. In Ref. [17] the wave functions were expanded in confinement eigenstates which are solutions of a relativistic Schrödinger equation with a confinement potential only. Here we choose cubic Hermite splines [18] as the basis functions. The cubic Hermite splines are piecewise polynomials of degree three with continuous first derivatives [18]. Also we impose the boundary conditions by requiring the auxiliary wave function be zero at $k=0$ and at a cutoff $k=k_{max}$. By expanding the wave function in N splines functions and requiring that the expansion satisfies Eq. (10) at N distinct values (collocation points) of k Eq. (10) can be converted to a matrix equation. The two-point Gaussian quadrature points on each interval are chosen as the collocation points. The eigenvalues and eigenvectors of the matrix equation are the masses of the mesons and the coefficients of the meson wave function bases respectively.

III. RESULTS

The light meson mass spectra can be obtained by solving for the eigenvalues of the relativistic Hamiltonian $H=T+V-M_{q\bar{q}}$ where V is given in Eq. (5) and $M_{q\bar{q}}$ is a constant

energy to be determined from data. If $M_{q\bar{q}} = 0$, the model gives the meson mass spectra; if $M_{q\bar{q}} \neq 0$, then the model only gives the splittings. In our opinion, this overall constant energy is somewhat artificial. In our first calculation, the parameters used are $M = 360$ MeV, $f_c\alpha_s = -0.5$ and $\sqrt{\sigma} = 0.197$ (GeV)^2. These parameters are chosen so that the results can be compared with the results of Ref. [10] which used a variational method to solve the wave equation in coordinate space. Unfortunately, they used a different method to treat the spin-orbit interaction so we can only directly compare our results with theirs without spin-orbit interaction, i.e. for π, ρ and b mesons. We initially set $C_L = C_C = C_{SS} = 1$, $C_{LS} = C_T = 0$ and eliminate the vertex form factor for the Coulomb potential. These results are listed in Table I with the same constant energy ($M_{q\bar{q}} = 750$ MeV) subtracted out.

Our results are consistently higher than their results by $25 - 60$ MeV [19]. We have been unable to establish the origin of the discrepancy but note that our Fortran codes were cross checked with a totally different method [20] and we agree within numerical error (i.e., < 0.5 MeV).

The results of fitting light meson spectra by varying two parameters Λ and C_{LS} ($C_L = C_C = C_{SS} = C_T = 1$, $M = 360$ MeV, no form factor for the Coulomb potential) are shown in Table II. The differences between this second calculation and that of Ref. [10] are due to the following factors:

a) We use $\Lambda = 0.6426$ GeV^{-1}; they used $\Lambda = 0.13$ fm $= 0.658805$ GeV^{-1}.

b) The spin-orbit interaction is treated differently and our strength of the spin-orbit interaction is $C_{LS} = 0.3236$.

c) We consider the coupling between different channels via the tensor interaction.

d) A different constant energy ($M_{q\bar{q}} = 738.24$ MeV) is used here versus 750 MeV.

The average deviation from the fifteen experimental values is 69 MeV for our results and 92 MeV for Ref. [10].

As we mentioned in the previous section, we are free to adjust all seven parameters (M, Λ, C_L, C_C ,C_{SS}, C_{LS}, C_T) to fit the light meson spectra. We also eliminate the need for the constant energy term $M_{q\bar{q}}$ used above and use a form factor not only for spin-spin, spin-orbit and tensor interaction but also for the Coulomb interaction. The parameters we use are $M = 258$ MeV, $\Lambda = 0.645$ GeV^{-1}, $C_L = 0.6704$, $C_C = 3.3824$, $C_{SS} = 0.155$, $C_{LS} = 0.0448$ and $C_T = 0.1868$. The results of this fitting are listed in the last column of Table II. The average deviation is 28 MeV for eleven states which can be compared to the average deviation of 55 MeV in a similar model calculated in Ref. [12]. The average deviation from all fifteen experimental values is 51 MeV which is significantly better than that obtained by only varying two parameters (69 MeV).

The above parameter values reflect the following: The linear confinement potential is a little weaker than for static quarks ($C_L \simeq 1$). The C_C value is considerably greater than one, which means that light quark systems have a running coupling constant ($\alpha_s C_C$) much larger than heavy quark systems as might be expected from QCD (asymptotic freedom). C_{LS} is very weak as observed by others [13]. C_{SS} and C_T are also reduced relative to values obtained for heavy quarks.

IV. FUTURE WORK

Potential models based on the constituent quark model have enjoyed considerable success. Even the nonrelativistic potential model works much better than one might naïvely expect. However, the nonrelativistic constituent quark model fails to explain the meson and baryon form factors at high energy because the calculated form factors at high energy fall off too fast. In our calculations, the tails of the momentum space wave functions have a power-law fall off, rather than the exponential decay of harmonic oscillator potential models. It is hoped that using a wave function like ours with an appropriate relativistic "boost" might be able to provide the correct form factor behavior at high energy. This question is currently being pursued.

In 1939, Wigner [21] showed that the description of a physical system in a relativistic quantum theory must correspond to a unitary representation of the Poincaré group. In 1961, Foldy [22] recognized the importance of macroscopic locality (cluster separability) as an additional constraint on relativistic potential models. Many "relativistic" three-body potential models [23] do not satisfy either of these two requirements. Relativistic three-body mass operators which satisfy both of these two requirements have been discussed in a recent review [5]. A relativistic three-body bound state formulation which incorporates Wigner's theorem and macroscopic locality can be solved by using the relativistic Faddeev equation for the eigenvalues and eigenvectors of the three-body mass operator in momentum space. The calculations are very complicated since the potential operators are functions of the position operators and are inside the square roots. These difficulties can be overcome by using relativistic two-body wave functions as the basis [7,8]. Therefore, solving the two-body equation accurately in momentum space was necessary as a first step. The two-body calculations discussed in this paper are important primarily because of their use as input for three-quark calculations. Our ultimate purpose is to solve the relativistic bound state problem for three light constituent quarks.

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TABLES

$nLSJ^\pi$	state	Exp. value	$E-750$	Ref. [10]
0000^-	π	138	164.8	140
0011^-	ρ	768.3	810.9	751
0101^+	b_1	1233	1155.0	1113
1000^-	$\pi(1300)$	1300	1143.2	1114
1011^-	$\rho(1450)$	1450	1490.6	1442
0202^-	π_2	1665	1603.5	1560
0303^+	b_3		1930.6	1892
2011^-	$\rho(1700)$	1700	1995.4	1957
0404^-	π_4		2214.0	2177
0505^-	b_5		2470.8	2438

TABLE I. The light meson mass spectrum without spin-orbit and tensor interactions. Our results (the fourth column) are consistently higher than those of Ref. [10] by 25-60 MeV.

$nLSJ^\pi$	state	Exp. value	Ref. [10]	$E-738.24$	Ref. [12]	E
0000^-	π	138	140	133.6	132	140.1
0011^-	ρ	768.3	751	807.2	755	775.7
0101^+	b_1	1233	1113	1165.0	1106	1174.6
0110^+	a_0	983.3	931	971.2	1108	973.7
0111^+	a_1	1260	1180	1251.7	1273	1298.2
0112^+	a_2	1318.4	1256	1300.2	1328	1323.0
1000^-	$\pi(1300)$	1300	1114	1140.8	1194	1188.9
1011^-	$\rho(1450)$	1450	1442	1473.3	1521	1472.7
0202^-	π_2	1665	1560	1615.7	1675	1661.9
0213^-	$\rho_3(1690)$	1691	1620	1668.0	1754	1702.2
0313^+	$a_3(2050)$	2050	1894	1944.3		1981.6
0314^+	$a_4(2040)$	2040	1933	1980.0	2100	2001.1
2011^-	$\rho(1700)$	1700	1957	2004.4		1960.5
0415^-	$\rho_5(2350)$	2350	2207	2257.7		2256.2
0516^+	$a_6(2350)$	2450	2462	2510.9		2483.1

TABLE II. The light meson mass spectrum by varying two parameters Λ and C_{LS} (the fifth column) and those by varying all 7 parameters (the last column). The average deviation from the fifteen experiment values is 69 MeV for the second calculation (the fifth column) and 51 MeV for the third calculation (the last column).